An Extended Reciprocally Convex Matrix Inequality for Stability Analysis of Systems with Time-Varying Delay ⋆

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Abstract

The reciprocally convex combination lemma (RCCL) is an important technique to develop stability criteria for the systems with a time-varying delay. This note develops an extended reciprocally convex matrix inequality, which reduces the estimation gap of the RCCL-based matrix inequality and reduces the number of decision variables of the recently proposed delay-dependent RCCL. A stability criterion of a linear time-delay system is established through the proposed matrix inequality. Finally, a numerical example is given to demonstrate the advantage of the proposed method.

Key words: Time-delay system, time-varying delay, extended reciprocally convex matrix inequality, stability

1 Introduction

Since time-varying delays arising in communication networks may result in instability phenomenon, the stability analysis of time-delay systems has become a hot topic in the past few decades [1]. The main attention is paid to determine the admissible delay region, for which the systems remain stable, by developing effective delay-dependent stability criteria via the Lyapunov-Krasovskii stability theory [2]. The difficulty relies on the handling of the integral terms arising in the derivative of the Lyapunov-Krasovskii functinals (LKFs) [3]. The development of new methods for this problem has always been an important consideration.

The model transformations [4] and the free-weighting-matrix approach [5] were applied to handle the integral terms in the early literature. In recent years, estimating integral terms directly via integral inequalities gradually becomes more popular [6]. Various integral inequalities have been developed, such as Jensen inequality [7], Wirtinger-based inequality [3], auxiliary function based inequalities [8–10], Bessel-Legendre inequality [11], and free-matrix-based inequalities [12,13]. For a system with constant delay, the Bessel-Legendre inequality has potential to provide the analytical solution.

During analyzing the systems with a time-varying delay via those inequalities, an additional technique is needed to deal with the time-varying delay arising in the estimated terms [14]. The simplest treatment (replacing time-varying delay with its bounds [15–17]) inevitably leads to the conservatism. The free-matrix-based inequality can lead to convex conditions, which can be treated by the convex combination technique [18], by adding many slack matrices. Furthermore, the reciprocally convex combination lemma (RCCL) can directly handle the time-varying delay just introducing a few slack matrices [19]. Thus, the RCCL, combined with integral inequalities, has become the most popular method to estimate the integral terms with time-varying delays. Two steps, bounding the terms by integral in-

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equalities and handling the time-varying delays, are required for this estimation task [24]. For the first step, the Bessel-Legendre inequality provides a unified way to reduce the estimation gap [11]. There seems no much room for further improvement. For the second step, the time-varying delays in the denominators is commonly treated through the RCCL [19]. The delay-dependent RCCL has improved the RCCL in the first time [22] and there still exists room to further improvement. This motivates the current research.

This note develops an extended reciprocally convex matrix inequality inspired by the work of [24,25]. Firstly, the theoretical analysis shows that the proposed inequality can not only directly lead to the inequalities reported in [24,25] but also reduce the estimation gap of the RCCL-based matrix inequality and the number of decision variables of the recently proposed delay-dependent RCCL. Then, the proposed inequality is applied to derive a new stability criterion of a linear time-delay system. Finally, the advantage of the proposed method is demonstrated through a numerical example.

Notations: Throughout this note, the superscripts T and -1 mean the transpose and the inverse of a matrix, respectively; $\mathcal{R}^n$ denotes the n-dimensional Euclidean space; $\|\cdot\|$ refers to the Euclidean vector norm; $col\{x_1,\cdots,x_n\}=[x_1^T,\cdots,x_n^T]^T$; $P>0$ (≥ 0) means P is a real symmetric and positive-definite (semi-positive-definite) matrix; I and 0 stand for the identity and the zero matrices, respectively; $diag\{\cdot\}$ denotes the block-diagonal matrix; and the symmetric term in the symmetric matrix is denoted by $*$.

2 Problem Formulation

Consider a linear system with a time-varying delay:

$$\begin{cases} \dot{x}(t)=Ax(t)+A_dx(t-d(t)),\ t\geq 0 \\ x(t)=\phi(t),\quad t\in[-h_2,0] \end{cases} \tag{1}$$

where $x(t)\in\mathcal{R}^n$ is the system state, A and A_d are the system matrices, the initial condition $\phi(t)$ is a continuously function, and the delay $d(t)$ satisfying

$$h_1\leq d(t)\leq h_2 \tag{2}$$

where h_1 and h_2 are constant. Let $h_{12}=h_2-h_1$.

The following lemmas will be applied in this note.

Lemma 1 *[11] For a symmetric matrix $R>0$, scalars a and b with $a<b$, and vector x such that the integrations concerned are well defined, the following inequality holds*

$$(b-a)\int_a^b \dot{x}^T(s)R\dot{x}(s)ds\geq\sum_{i=1}^{N}(2i-1)\chi_i^TR\chi_i \tag{3}$$

where $\chi_1=x(b)-x(a)$, $\chi_2=x(b)+x(a)-\frac{2}{b-a}\int_a^b x(s)ds$, $\chi_3=x(b)-x(a)+\frac{6}{b-a}\int_a^b x(s)ds-\frac{12}{(b-a)^2}\int_a^b\int_s^b x(u)duds$. *(Since only $N=2,3$ will be used in this note, $\chi_i, i\geq 4$ that can be found from [11] are omitted in this note.)*

Lemma 2 *[21] For a symmetric matrix $R>0$, scalars a and b with $a<b$, and vector x such that the integrations concerned are well defined, the following inequality holds*

$$\begin{aligned}\frac{(b-a)^2}{2}&\int_a^b\int_\theta^b x^T(s)Rx(s)dsd\theta \\ &\geq\Big(\int_a^b\int_\theta^b x(s)dsd\theta\Big)^TR\Big(\int_a^b\int_\theta^b x(s)dsd\theta\Big)\end{aligned} \tag{4}$$

Lemma 3 *(Delay-dependent RCCL [22]) For a real scalar $\alpha\in(0,1)$, symmetric matrices $X_1>0$ and $X_2>0$, and any matrices S_1, S_2, U_1, and U_2, if the following inequality holds*

$$\begin{bmatrix} X_1 & 0\\ 0 & X_2\end{bmatrix}-\alpha\begin{bmatrix} U_2 & S_2\\ * & 0\end{bmatrix}-(1-\alpha)\begin{bmatrix} 0 & S_1\\ * & U_1\end{bmatrix}\geq 0 \tag{5}$$

then the following matrix inequality holds

$$\begin{bmatrix} \frac{X_1}{\alpha} & 0\\ 0 & \frac{X_2}{1-\alpha}\end{bmatrix}\geq\begin{bmatrix} X_1 & 0\\ 0 & X_2\end{bmatrix}+(1-\alpha)\begin{bmatrix} U_2 & S_1\\ * & 0\end{bmatrix}+\alpha\begin{bmatrix} 0 & S_2\\ * & U_1\end{bmatrix} \tag{6}$$

3 An extended matrix inequality

Lemma 4 *For a real scalar $\alpha\in(0,1)$, symmetric matrices $X_1>0$ and $X_2>0$, and any matrices S_1 and S_2, the following matrix inequality holds*

$$\begin{bmatrix} \frac{1}{\alpha}X_1 & 0\\ 0 & \frac{1}{1-\alpha}X_2\end{bmatrix}\geq\begin{bmatrix} X_1+(1-\alpha)T_1 & (1-\alpha)S_1+\alpha S_2\\ * & X_2+\alpha T_2\end{bmatrix} \tag{7}$$

where $T_1=X_1-S_2X_2^{-1}S_2^T$ and $T_2=X_2-S_1^TX_1^{-1}S_1$.

Proof: It follows $X_i\geq 0, i=1,2$ and Schur complement that

$$\Lambda_1=\begin{bmatrix} X_1 & S_1\\ * & S_1^TX_1^{-1}S_1\end{bmatrix}\geq 0,\Lambda_2=\begin{bmatrix} S_2X_2^{-1}S_2^T & S_2\\ * & X_2\end{bmatrix}\geq 0 \tag{8}$$

Let ξ_a and ξ_b be two any vectors of $\mathcal{R}^m$, set $g_1=\sqrt{(1-\alpha)/\alpha}$, $g_2=-\sqrt{\alpha/(1-\alpha)}$, and define

$$\Theta_1(\alpha)=\begin{bmatrix}\xi_a\\ \xi_b\end{bmatrix}^T\begin{bmatrix} \frac{1}{\alpha}X_1 & 0\\ 0 & \frac{1}{1-\alpha}X_2\end{bmatrix}\begin{bmatrix}\xi_a\\ \xi_b\end{bmatrix}$$

$$\Theta_2(\alpha)=\begin{bmatrix}\xi_a\\ \xi_b\end{bmatrix}^T\begin{bmatrix} X_1 & (1-\alpha)S_1+\alpha S_2\\ * & X_2\end{bmatrix}\begin{bmatrix}\xi_a\\ \xi_b\end{bmatrix}$$

$$\Theta_3(\alpha)=\begin{bmatrix}\xi_a\\ \xi_b\end{bmatrix}^T\begin{bmatrix}(1-\alpha)T_1 & 0\\ 0 & \alpha T_2\end{bmatrix}\begin{bmatrix}\xi_a\\ \xi_b\end{bmatrix}$$

Carrying out simple calculations and using (8) yields

$$\begin{aligned}&\Theta_1(\alpha)-\Theta_2(\alpha)-\Theta_3(\alpha)\\ =&\begin{bmatrix}\xi_a\\ \xi_b\end{bmatrix}^T\left\{\begin{bmatrix}\frac{X_1}{\alpha} & 0\\ 0 & \frac{X_2}{1-\alpha}\end{bmatrix}-\begin{bmatrix}X_1+(1-\alpha)T_1 & (1-\alpha)S_1+\alpha S_2\\ * & X_2+\alpha T_2\end{bmatrix}\right\}\begin{bmatrix}\xi_a\\ \xi_b\end{bmatrix}\\ =&\begin{bmatrix}g_1\xi_a\\ g_2\xi_b\end{bmatrix}^T\begin{bmatrix}X_1-\alpha T_1 & (1-\alpha)S_1+\alpha S_2\\ * & X_2-(1-\alpha)T_2\end{bmatrix}\begin{bmatrix}g_1\xi_a\\ g_2\xi_b\end{bmatrix}\\ =&\begin{bmatrix}g_1\xi_a\\ g_2\xi_b\end{bmatrix}^T[(1-\alpha)\Lambda_1+\alpha\Lambda_2]\begin{bmatrix}g_1\xi_a\\ g_2\xi_b\end{bmatrix}\\ \geq& 0\end{aligned}$$

The above inequality holds for all vectors ξ_a and ξ_b, which implies (7). This completes the proof. ■

Remark 5 *Compared with the inequalities developed in [24,25], the matrix inequality (7) has two advantages.*

- *The inequalities in [24,25] can be directly obtained via the proposed matrix inequality (7). For example, Lemma 4 of [24] can be obtained by using the Wirtinger-based integral inequality and (7) with $S_1 = S_2$, i.e.,*

$$\begin{aligned}h\int_{t-h}^{t}\dot{x}^T(s)R\dot{x}(s)ds\geq&\begin{bmatrix}\xi_1(t)\\ \xi_2(t)\end{bmatrix}^T\begin{bmatrix}\frac{h\tilde{R}}{d(t)} & 0\\ 0 & \frac{h\tilde{R}}{h-d(t)}\end{bmatrix}\begin{bmatrix}\xi_1(t)\\ \xi_2(t)\end{bmatrix}\\ \geq&\begin{bmatrix}\xi_1(t)\\ \xi_2(t)\end{bmatrix}^T\begin{bmatrix}\tilde{R}+\frac{h-d(t)}{h}\tilde{T}_1 & S\\ * & \tilde{R}+\frac{d(t)}{h}\tilde{T}_2\end{bmatrix}\begin{bmatrix}\xi_1(t)\\ \xi_2(t)\end{bmatrix}\end{aligned}$$

  Similarly, Lemma 6 of [24] (or Lemma 4 of [25]) can be obtained by using (7) with $S_1 = S_2$ and inequality (3) with $N = 3$ (or Wirtinger-based summation inequality [26]). Obviously, the development of those inequalities through (7) can avoid the complex proof (defining auxiliary functions and setting special matrices, etc.) required in [24,25].
- *What is more, matrix inequality (7) can be easily combined with the recently proposed integral/summation inequalities [10,11,27,28] to develop new inequalities with less estimation gaps. For example, it is predictable that the integral inequalities obtained by combining (7) and inequality (3) with $N > 3$ are less conservative than the ones in [24]. Moreover, there is an case that inequality (7) can deal with while the ones in [24] cannot do (see Remark 9 for details).*

Remark 6 *The proposed matrix inequality (7) improves the popular RCCL-based matrix inequality [20].*

- *At first, the proposed matrix inequality (7) includes the following inequality as a special case:*

$$\begin{bmatrix}\frac{1}{\alpha}X_1 & 0\\ 0 & \frac{1}{1-\alpha}X_2\end{bmatrix}\geq\begin{bmatrix}X_1+(1-\alpha)T_3 & S\\ * & X_2+\alpha T_4\end{bmatrix}\quad(9)$$

  where $T_3 = X_1 - SX_2^{-1}S^T$ and $T_4 = X_2 - S^TX_1^{-1}S$. Obviously, inequality (7) has less conservative than (9) due to no requirement of $S = S_1 = S_2$.
- *Secondly, by introducing the T_3- and T_4-dependent extra terms and waiving the requirement of $\begin{bmatrix}X_1 & S\\ * & X_2\end{bmatrix}\geq 0$, inequality (9) can reduce the estimation gap of the following popular RCCL-based inequity [20]*

$$\begin{bmatrix}\frac{1}{\alpha}X_1 & 0\\ 0 & \frac{1}{1-\alpha}X_2\end{bmatrix}\geq\begin{bmatrix}X_1 & S\\ * & X_2\end{bmatrix}\quad(10)$$

  where $\begin{bmatrix}X_1 & S\\ * & X_2\end{bmatrix}\geq 0$ with any matrix S.

Therefore, the proposed inequality (7) can reduce the estimation gap of the RCCL-based matrix inequality (10).

Remark 7 *Compared with the delay-dependent RCCL (6), the proposed inequality (7) with less number of decision variables included can provide the same estimation gap. Let J_1 and J_2 be the estimation gap of (6) and (7), respectively, then the difference of them is given as*

$$J_1-J_2=\begin{bmatrix}(1-\alpha)(X_1-S_2X_2^{-1}S_2^T-U_2) & 0\\ 0 & \alpha(X_2-S_1^TX_1^{-1}S_1-U_1)\end{bmatrix}$$

- *On the one hand, if the free matrices, U_1 and U_2, in (6) are optimally choose to be $X_2 - S_1^TX_1^{-1}S_1$ and $X_1 - S_2X_2^{-1}S_2^T$, then two inequalities have the same estimation gap ($J_1 = J_2$).*
- *On the other hand, condition (5) can be rewritten as*

$$\alpha\begin{bmatrix}X_1-U_2 & -S_2\\ * & X_2\end{bmatrix}+(1-\alpha)\begin{bmatrix}X_1 & -S_1\\ * & X_2-U_1\end{bmatrix}\geq 0\quad(11)$$

  which requires $\begin{bmatrix}X_1-U_2 & -S_2\\ * & X_2\end{bmatrix}\geq 0$ and $\begin{bmatrix}X_1 & -S_1\\ * & X_2-U_1\end{bmatrix}\geq 0$, which implies $X_1 - U_2 - S_2X_2^{-1}S_2^T \geq 0$ and $X_2 - U_1 - S_1^TX_1^{-1}S_1 \geq 0$. Then, $J_1 - J_2 \geq 0$ for all U_1 and U_2. That is, no matter how to optimize free matrices, U_1 and U_2, in (6), the estimation gap of (6) will not be less than that of (7).

Therefore, (6) and (7) have the same estimation gap.

4 A stability criterion

For system (1) with the delay satisfying (2), the following criterion is obtained via the proposed inequality (7).

Theorem 8 *For given scalars h_1 and h_2, system (1) with a time-varying delay satisfying (2) is asymptotically stable if there exist a $3n\times 3n$ matrix $P>0$, $n\times n$ matrices $Q_i>0$, $i=1,2$, $Z>0$, $R>0$, and $U>0$, and $2n\times 2n$ matrices S_1 and S_2, such that*

$$\Phi_4=\begin{bmatrix}\Psi_{1,[h_1]}-\Psi_2-\Psi_4 & E_2^TS_2\\ * & -\tilde{R}\end{bmatrix}<0 \tag{12}$$

$$\Phi_5=\begin{bmatrix}\Psi_{1,[h_2]}-\Psi_2-\Psi_5 & E_3^TS_1^T\\ * & -\tilde{R}-\tilde{U}\end{bmatrix}<0 \tag{13}$$

where

$$\begin{aligned}
\Psi_{1,[d(t)]}&=\Pi_1^TP\Pi_2+\Pi_2^TP\Pi_1+e_1^TQ_1e_1+e_2^T(Q_2-Q_1)e_2\\
&\quad -e_4^TQ_2e_4+e_s^T[h_1^2Z+h_{12}^2(R+U/2)]e_s\\
\Psi_2&=E_1^T[diag\{Z,3Z\}]E_1+2E_4^TUE_4+2E_5^TUE_5\\
\Psi_4&=\begin{bmatrix}E_2\\E_3\end{bmatrix}^T\begin{bmatrix}2\tilde{R}+\tilde{U} & S_1\\ * & \tilde{R}\end{bmatrix}\begin{bmatrix}E_2\\E_3\end{bmatrix}\\
\Psi_5&=\begin{bmatrix}E_2\\E_3\end{bmatrix}^T\begin{bmatrix}\tilde{R} & S_2\\ * & 2\tilde{R}\end{bmatrix}\begin{bmatrix}E_2\\E_3\end{bmatrix}\\
\Pi_1&=col\,\{e_1,h_1e_5,(d(t)-h_1)e_6+(h_2-d(t))e_7\}\\
\Pi_2&=col\,\{e_s,e_1-e_2,e_2-e_4\}\\
E_i&=col\,\{e_i-e_{i+1},\ e_i+e_{i+1}-2e_{i+4}\},i=1,2,3\\
E_4&=e_2-e_6,\quad E_5=e_3-e_7\\
e_s&=[A,0,A_d,0,0,0,0],\\
e_i&=[0_{n\times(i-1)n},I,0_{n\times(7-i)n}],i=1,2,\cdots,7\\
\tilde{R}&=diag\{R,3R\},\quad \tilde{U}=diag\{U,3U\}
\end{aligned}$$

Proof: Construct the following LKF candidate:

$$\begin{aligned}
V(x_t,\dot{x}_t)&=\eta^T(t)P\eta(t)\\
&+\int_{t-h_1}^{t}x^T(s)Q_1x(s)ds+h_1\int_{-h_1}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)Z\dot{x}(s)dsd\theta\\
&+\int_{t-h_2}^{t-h_1}x^T(s)Q_2x(s)ds+h_{12}\int_{-h_2}^{-h_1}\int_{t+\theta}^{t}\dot{x}^T(s)R\dot{x}(s)dsd\theta\\
&+\int_{-h_2}^{-h_1}\int_{s}^{-h_1}\int_{t+\theta}^{t}\dot{x}^T(u)U\dot{x}(u)dud\theta ds
\end{aligned}\tag{14}$$

where $P>0$, $Q_i>0$, $Z>0$, $R>0$, and $U>0$; $\eta(t)=col\{x(t),\int_{t-h_1}^{t}x(s)ds,\int_{t-h_2}^{t-h_1}x(s)ds,\int_{-h_1}^{0}\int_{t+\theta}^{t}x(s)dsd\theta\}$. It is easily found that the LKF satisfies $V(x_t,\dot{x}_t)\geq\epsilon||x(t)||^2$ for a sufficient small scalar $\varepsilon>0$.

Calculating the derivative of $V(x_t,\dot{x}_t)$ along the solution of system (1) yields

$$\begin{aligned}
\dot{V}(x_t,\dot{x}_t)&=\zeta^T(t)\Psi_{1,[d(t)]}\zeta(t)-h_1\int_{t-h_1}^{t}\dot{x}^T(s)Z\dot{x}(s)ds\\
&-h_{12}\int_{t-h_2}^{t-h_1}\dot{x}^T(s)R\dot{x}(s)ds\\
&-\int_{-d(t)}^{-h_1}\int_{t+\theta}^{t-h_1}\dot{x}^T(s)U\dot{x}(s)dsd\theta\\
&-\int_{-h_2}^{-d(t)}\int_{t+\theta}^{t-d(t)}\dot{x}^T(s)U\dot{x}(s)dsd\theta\\
&-(h_2-d(t))\int_{t-d(t)}^{t-h_1}\dot{x}^T(s)U\dot{x}(s)dsd\theta
\end{aligned}\tag{15}$$

By using (3) with $N=2$ and (4) to estimate the single integral and the double integral terms, respectively, $\dot{V}(x_t,\dot{x}_t)$ can be estimated as

$$\begin{aligned}
\dot{V}_1(x_t,\dot{x}_t)&\leq\zeta^T(t)[\Psi_{1,[d(t)]}-\Psi_2+E_2^T\tilde{U}E_2]\zeta(t)\\
&-\zeta^T(t)\left[\frac{h_{12}E_2^T(\tilde{R}+\tilde{U})E_2}{d(t)-h_1}+\frac{h_{12}E_3^T\tilde{R}E_3}{h_2-d(t)}\right]\zeta(t)
\end{aligned}$$

where $\zeta(t)=col\,\{x(t),x(t-h_1),x(t-d(t)),x(t-h_2),\int_{t-h_1}^{t}\frac{x(s)}{h_1}ds,,\int_{t-d(t)}^{t-h_1}\frac{x(s)}{d(t)-h_1}ds,\int_{t-h_2}^{t-d(t)}\frac{x(s)}{h_2-d(t)}ds\}$.

For matrices S_1 and S_2, it follows (7) that

$$\begin{aligned}
&\zeta^T(t)\left[\frac{h_{12}E_2^T(\tilde{R}+\tilde{U})E_2}{d(t)-h_1}+\frac{h_{12}E_3^T\tilde{R}E_3}{h_2-d(t)}-E_2^T\tilde{U}E_2\right]\zeta(t)\\
&\geq\zeta^T(t)\bar{\Psi}_{3,[d(t)]}\zeta(t)
\end{aligned}\tag{16}$$

where

$$\begin{aligned}
\bar{\Psi}_{3,[d(t)]}&=E_2^T(\tilde{R}+\tilde{U})E_2+E_3^T\tilde{R}E_3-E_2^T\tilde{U}E_2\\
&+\frac{h_2-d(t)}{h_{12}}\begin{bmatrix}E_2\\E_3\end{bmatrix}^T\begin{bmatrix}\tilde{R}+\tilde{U}-S_2\tilde{R}^{-1}S_2^T & S_1\\ * & 0\end{bmatrix}\begin{bmatrix}E_2\\E_3\end{bmatrix}\\
&+\frac{d(t)-h_1}{h_{12}}\begin{bmatrix}E_2\\E_3\end{bmatrix}^T\begin{bmatrix}0 & S_2\\ * & \tilde{R}-S_1^T(\tilde{R}+\tilde{U})^{-1}S_1\end{bmatrix}\begin{bmatrix}E_2\\E_3\end{bmatrix}
\end{aligned}$$

Therefore, the derivative of $V(x_t,\dot{x}_t)$ is estimated as

$$\dot{V}(x_t,\dot{x}_t)\leq\zeta^T(t)\left[\Psi_{1,[d(t)]}-\Psi_2-\bar{\Psi}_{4,[d(t)]}\right]\zeta(t) \tag{17}$$

Based on convex combination technique, $\Psi_{1,[d(t)]}-\Psi_2-\bar{\Psi}_{3,[d(t)]}\leq 0$ holds if the following two inequalities hold

$$\Psi_{1,[h_1]}-\Psi_2-\bar{\Psi}_{3,[h_1]}<0 \tag{18}$$

$$\Psi_{1,[h_2]}-\Psi_2-\bar{\Psi}_{3,[h_2]}<0 \tag{19}$$

which are respectively guaranteed by (12) and (13) based on Schur complement. Thus, if (12) and (13) holds, then, for a sufficient small scalar $\varepsilon > 0$, $\dot{V}(x_t, \dot{x}_t) \leq -\varepsilon||x(t)||^2$ holds, which ensures that system (1) with the delay satisfying (2) is asymptotically stable. ■

Remark 9 *The single integral terms with the time-varying delay in (15) are summarized as*

$$h_{12}\int_{t-h_2}^{t-d(t)} \dot{x}^T(s)R\dot{x}(s)ds + h_{12}\int_{t-d(t)}^{t-h_1} \dot{x}^T(s)\breve{R}\dot{x}(s)dsd\theta$$

where $\breve{R} = R + \frac{h_2-d(t)}{h_{12}}U$. Since the integral inequalities reported in [24] require the same integrand of two integral terms, they are no longer available to directly estimate the above terms due to $R \neq \breve{R}$. It shows the advantage of the proposed matrix inequality (7) in comparison with the integral inequalities in [24].

Remark 10 *Although this note only discusses the time delay satisfying (2), it is predictable that the proposed R-CCL can be also used to improve the delay-rate-dependent stability criterion when the delay change rate is available. Moreover, the proposed RCCL can be combined with more effective LKFs to further improve the results.*

5 A numerical example

A numerical example is used to demonstrate the advantages of the proposed matrix inequality.

Example 1 *Consider system (1) with the parameters*

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \tag{20}$$

The maximal upper bounds of h_2 with respect to various h_1 calculated by Theorem 1 and reported in the typical literature are listed in Table 1, where Th. and Co. respectively indicate Theorem and Corollary, and the number of decision variables (NoVs) are also given. Compared with the early results (such as the ones based on free-weighting-matrix approach [15], and Jensen inequality [16,17,19,23]), Theorem 8 obviously reduce the conservatism with the reasonable increase of NoVs. Theorem 8 achieves the reduction of both the conservatism and the NoVs in comparison with the auxiliary-function-based inequality [8,9]. Moreover, compared with the criterion obtained by the delay-dependent RCCL [22], Theorem 8 provides the same results but requires less NoVs.

6 Conclusions

This note has developed an extended reciprocally convex matrix inequality for the systems with a time-varying delay. It has been found theoretically that the proposed matrix inequality reduces the estimation gap of the popular RCCL-based matrix inequality and that it provides the same estimation gap with less decision variables required in comparison with the recently proposed delay-dependent RCCL. Thus, the proposed matrix inequality has potential to improve the original/delay-dependent RCCL based criteria by combining it with various integral/summation inequalities. For a continuous linear system with a time delay, a new stability criterion has been established via the developed matrix inequality, and a numerical example has been given to show the advantage of the proposed method.

Table 1
The maximal upper bounds of h_2 for various h_1.

Methods	h_1				NoVs
	0	0.4	0.7	1.0	
Co.6 [15]	1.345	1.440	1.573	1.742	$8.5n^2 + 2.5n$
Co.2 [16]	1.529	1.619	1.729	1.873	$2.5n^2 + 2.5n$
Co.7 [17]	1.529	1.625	1.743	1.900	$17.5n^2 + 7.5n$
Th.2 [19]	1.868	1.882	1.953	2.066	$3.5n^2 + 2.5n$
Co.2 [23]	1.868	1.89	1.98	2.120	$11n^2 + 3n$
Th.1 [20]	2.113	2.179	2.237	2.318	$10.5n^2 + 3.5n$
Th.2 [9]	2.113	2.180	2.237	2.319	$19.5n^2 + 4.5n$
Th.1 [8]	2.14	2.19	2.24	2.31	$21n^2 + 6n$
Th.1 [22]	2.213	2.256	2.286	2.345	$18.5n^2 + 5.5n$
Th.7	2.213	2.256	2.286	2.345	$15n^2 + 4n$

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